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On the Range of Validity of a Simple Wave Approximation of a Non-Linear Set of Diffusive Wave Equations

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SUMMARY

The set of wave equations considered is an intermediate approximation of the Navier–Stokes equations. A further approximation leads to Burgers' equation. The range of validity of this simple wave approximation has been studied. The method used is especially useful for small nonlinearity.

1. Introduction

In some preceding papers [1, 2], L. J. F. Broer and the present author have paid attention to the validity of an approximation method which applied to a certain class of initial value problems for the set

$$\alpha_t + [1+\varepsilon\phi(\alpha,\beta)]\alpha_x = \mu(\alpha_{xx}-\beta_{xx}),$$
$$\beta_t - [1+\varepsilon\phi(\alpha,\beta)]\beta_x = \mu(\beta_{xx}-\alpha_{xx}),$$

where ε and μ are positive constants, the subscript x (or t) denotes partial differentiation with respect to x (or t) and, if $\mu=0$, the remaining set is hyperbolic.

In [1], this has been done for a linear set of equations ($\varepsilon=0$) by making use of the explicit solution and in [2] for a set which is, as $\mu=0$, totally exceptional in the sense of Lax [3]. The latter equations could be transformed into the linear equations studied in [1]. This was done by means of a nonlinear transformation. In both cases, the solutions of the equations did not contain shock waves. In this paper, we shall deal with equations that do have solutions of that kind.

In the hierarchy of approximations emanating from the Navier-stokes equations, Lighthill [4] finds the set

$$a_t + va_x + \frac{\gamma-1}{2}av_x = 0\,, \tag{1}$$

$$v_t + vv_x + \frac{2}{\gamma-1}aa_x = \delta v_{xx}\,, \tag{2}$$

where a is the sound velocity, v the flow velocity, $\gamma = C_p/C_v$ and δ is the diffusity of sound. We have

$$\delta = \tfrac{4}{3}\nu + \frac{\mu_v}{\rho_0} + \gamma - 1\,\frac{k}{\rho_0 C_p}\,,$$

where ρ is the density, ν the kinematic-, μ_v the bulk viscosity and k the coefficient of heat conduction. The subscript zero refers to the undisturbed situation.

The left hand side of equations (1) and (2) are the exact forms of the equations for sound waves of finite amplitude under thermodynamically reversible conditions. They form the basis of Riemann's classic analysis [5]. The right hand side consists of a linearized approxima-

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tion of the diffusion and heat conducting effects, obtained by assuming that δ and the dimensionless velocities $a-a_0/a_0$ and v/a_0 are small.

We shall not concern ourselves with the validity of the approximations Lighthill used to arrive at (1) and (2) but assume that, if δ and the dimensionless velocities are small enough, (1) and (2) describe a real physical situation.

By introducing the Riemann variables

$$\alpha = \tfrac{1}{2}v + \frac{a}{\gamma-1},$$

$$\beta = -\tfrac{1}{2}v + \frac{a}{\gamma-1},$$

we find

$$\alpha_t + \left[\frac{\gamma+1}{2}\alpha + \frac{\gamma-3}{2}\beta\right]\alpha_x = \tfrac{1}{2}\delta(\alpha_{xx}-\beta_{xx}), \tag{3}$$

$$\beta_t - \left[\frac{\gamma+1}{2}\beta + \frac{\gamma-3}{2}\alpha\right]\beta_x = \tfrac{1}{2}\delta(\beta_{xx}-\alpha_{xx}). \tag{4}$$

These equations are of the required form.

The approximation we shall consider and which is due to Lighthill, has been described extensively in [1] and [2]. Here we shall only give a brief account of the ideas behind it. The approximation applies to the class of initial conditions

$$\alpha(x, 0) = f(x),$$
$$\beta(x, 0) = \beta_0.$$

If $\delta = 0$, then (4) is satisfied identically and the solution of (3) is a simple wave. Now, the approximation which, as in [1] and [2], will be called the simple wave (sw) approximation henceforth, is based on the following assumption. When δ is small but not zero, then, for some finite interval of time, β will be negligible and α will be approximately described by the solution of an equation of Burgers type:

$$\alpha_t + \left[\frac{\gamma+1}{2}\alpha + \frac{\gamma-3}{2}\beta_0\right]\alpha_x = \tfrac{1}{2}\delta\alpha_{xx}. \tag{5}$$

However, the problem is that $\beta-\beta_0$ will grow from zero and therefore, it is not clear a priori that α satisfies (5) for longer intervals of time too. In this paper we shall indicate a range of validity of this sw approximation. The equation of Burgers, which is exactly solvable ([4, 6, 7]), is often used to describe the behaviour of small amplitude shock waves. Therefore, it is especially interesting to know whether or not the sw approximation holds, in a sense yet to be defined, in an interval of time larger than that necessary for a shock wave to develop.

In sections 2 and 3 some mathematical notation needed and the definition of what we shall call a good sw approximation is given. The method we shall follow to deal with the problem is explained in section 4. It is based on a priori estimates. In section 5 local a priori estimates are constructed. From these estimates, we obtain an upper bound for the range of values of t, for which the sw approximation holds. That upper bound is "always" smaller than T_{crit}, the time at which a shock wave starts to develop. Partly this is due to the method followed. In section 6, using global a priori estimates for Burgers' equation, we shall deal with the question whether or not it will be possible to improve the results found in section 5, in this way.

2. Mathematical Notations

R is the interval of the real numbers. T and N are positive numbers. Q_T^N is the rectangular domain of points x, t satisfying $0 \leq t \leq T$, $|x| \leq N$. As a rule the index N will be omitted. If

$N=\infty$, Q_T is denoted by H_T. Γ_T is that part of the boundary of Q_T consisting of the line segments $t=0$, $x=-N$ and $x=N$. $L_2(R)$ is the Hilbert-space consisting of all real square (Lebesque) integrable functions. The inner product (,) and norm $\| \ \|$ are defined by

$$(u, v)=\int_{-\infty}^{\infty} u(x)v(x)dx\,, \qquad \|u\|=(u, u)^{\frac{1}{2}}\,.$$

$W_2^n(R)$ is the Hilbert-space consisting of all elements of $L_2(R)$ having generalized derivatives up to order n inclusively, that are square integrable on R. The inner product $(\ ,\ )_n$ and norm $\| \ \|_n$ are defined by

$$(u, v)_n = \sum_{i=1}^{n} (D^i u, D^i v)+(u, v)\,, \qquad \|u\|_n=(u, u)_n^{\frac{1}{2}}\,,$$

where $D^i u$ is the generalized derivative of order i.

Introduce the following distance in Q_T:

$$d(P_1, P_2)=(|x'-x''|^2+|t'-t''|)^{\frac{1}{2}}\,,$$

where $P_1=(t', x')$ and $P_2=(t'', x'')$.

Let

$$|u|_0=\sup_{Q_T}|u|\,, \quad |u|_\alpha=|u|_0+\sup_{Q_T}\frac{|u(P_1)-u(P_2)|}{d(P_1, P_2)^\alpha}\,, \qquad 0<\alpha<1\,,$$

$$|u|_{1+\alpha}=|u|_\alpha+|u_x|_\alpha\,,$$

$$|u|_{2+\alpha}=|u|_{1+\alpha}+|u_t|_\alpha+|u_x|_{1+\alpha}\,.$$

$C^{2+\alpha}(Q_T)$ is the Banach-space consisting of all functions u on Q_T for which $|u|_{2+\alpha}<\infty$. The norm is defined by $|u|_{2+\alpha}$ (cf. Friedman [8]).

Consider in H_T a quasi-linear system of the form

$$\boldsymbol{L}\boldsymbol{u}\equiv \boldsymbol{u}_t-A\boldsymbol{u}_{xx}+B(\boldsymbol{u})\boldsymbol{u}_x=\boldsymbol{0}\,,$$

in which $\boldsymbol{u}(x, t)=(u_1(x, t), \ldots, u_n(x, t))$ is an unknown vector function, A is a constant, non-negative $n\times n$-matrix and B a $n\times n$-matrix of which the elements depend on $\boldsymbol{u}$.

Definition. *In H_T, a classical solution of the Cauchy problem*

$$\boldsymbol{L}\boldsymbol{u}=\boldsymbol{0}\,, \tag{1}$$

$$\boldsymbol{u}(x, 0)=\boldsymbol{f}(x)\,, \tag{2}$$

is a solution that is continuous in H_T, that has continuous derivatives $\boldsymbol{u}_t$, $\boldsymbol{u}_x$ and $\boldsymbol{u}_{xx}$ and satisfies (1) *at all interior points of H_T, that remains bounded as $|x|\to\infty$ and for which* (2) *is valid* (*cf.* [11]).

Finally we state two lemmas that will be used in the sequel.

Lemma 1. *Let $u\in W_2^n(R)$, then $D^j u\to 0$ as $|x|\to\infty$, where $j=0, 1, \ldots, n-1$.*

A proof may be found in Smirnow [9], p. 486. From Peletier and Wessels [10], we infer

Lemma 2. *Let $u\in W_2^1(R)$, then a continuous function $\tilde{u}$ exists with $\tilde{u}=u$ a.e. and*

$$\sup_{x\in R}|\tilde{u}(x)|\leqq \tfrac{1}{2}\sqrt{2}\,\|u\|_1\,.$$

The lemma is known as Sobolev's first embedding theorem.

3. The Definition of a Good SW Approximation

First, we shall write equations (1.3) and (1.4) in dimensionless form. Assume $3\geqq\gamma>1$ (Air $\gamma\simeq 1{,}4$). Introduce

$$x = Lx', \; t = La_0^{-1}t', \qquad \delta = 2a_0 L\mu,$$

$$\alpha = \frac{a_0}{\gamma-1} + \frac{2\varepsilon a_0}{\gamma+1}\alpha', \qquad \beta = \frac{a_0}{\gamma-1} + \frac{2\varepsilon a_0}{\gamma+1}\beta',$$

where L is some reference length connected with $\alpha'(x', 0)$, ε a dimensionless measure for the strength of the wave. ε and L will be chosen such that the absolute maximum of $\alpha'(x', t) + \beta'(x', t')$ is not larger than, and of $(d\alpha'/dx')(x', 0)$ is equal to one. It is not a priori clear that this is possible for all $t \geqq 0$. However, it will turn out to be possible for the range of t-values we are interested in.

Performing the indicated substitutions in (1.3) and (1.4) and dropping the accents, we obtain

$$\alpha_t + [1 + \varepsilon\alpha + \varepsilon\theta\beta]\alpha_x = \mu(\alpha_{xx} - \beta_{xx}), \tag{1}$$

$$\beta_t - [1 + \varepsilon\beta + \varepsilon\theta\alpha]\beta_x = \mu(\beta_{xx} - \alpha_{xx}), \tag{2}$$

where $\theta = (\gamma-3)/(\gamma+1)$ $(-1 < \theta \leqq 0)$.

The initial conditions become

$$\alpha(x, 0) = f(x), \tag{3}$$

$$\beta(x, 0) = 0. \tag{4}$$

The sw approximation is given by the solution α_0 (here and in the following, the subscript zero no longer denotes the undisturbed value of a quantity) of

$$\alpha_{0t} + [1 + \varepsilon\alpha_0]\alpha_{0x} = \mu\alpha_{0xx}, \tag{5}$$

$$\alpha_0(x, 0) = f(x). \tag{6}$$

Definition. *For solutions α and α_0, both belonging to $L_2(R)$ (these are the only ones we consider here), α_0 will be called a good sw approximation of α in the interval of time $[0, T]$ if, for all $t \in [0, T]$*:

$$||\alpha - \alpha_0|| \leqq \delta ||f|| \qquad (0 < \delta < 1). \tag{7}$$

δ is a measure for the deviation of α_0 from α. Let T_m be the largest value of T for which (7) still holds. Our problem will be to find an estimate for T_m in terms of ε, μ, θ and the initial condition $f(x)$.

Finally we remark that this definition of a good sw approximation is weaker than that used in [1] and [2]. There, (7) has been replaced by $||\alpha - \alpha_0|| \ll ||\alpha||$.

4. Method of Solution

From now on, speaking about α, β and α_0, we shall mean the classical solution of (3.1), ..., (3.4), respectively (3.5) and (3.6).

Assume that, for all $t \in [0, T]$ (in this and the next sections we assume $T \geqq T_m$), α, β and α_0 belong to $W_2^2(R)$. Then according to (3.1), (3.2), (3.5) and lemma 2, for all $t \in [0, T]$, α_t and α_{0t} are in $L_2(R)$ too. Subtracting (3.5) from (3.1), multiplying the resulting equation by $\alpha - \alpha_0$ and integrating with respect to x over the entire x-axis, we find:

$$\frac{d}{dt}||\alpha - \alpha_0||^2 + \int_{-\infty}^{\infty} \frac{\partial}{\partial x}(\alpha - \alpha_0)^2 dx + 2\varepsilon \int_{-\infty}^{\infty} (\alpha - \alpha_0)(\alpha\alpha_x - \alpha_0\alpha_{0x}) dx +$$

$$+ 2\varepsilon\theta \int_{-\infty}^{\infty} (\alpha - \alpha_0)\beta\alpha_x dx = 2\mu \int_{-\infty}^{\infty} (\alpha - \alpha_0)(\alpha - \alpha_0 - \beta)_{xx} dx.$$

The interchangement of differentiation with respect to t and integration with respect to x was allowed as, for all $t \in [0, T]$, $\alpha - \alpha_0$ and $\alpha_t - \alpha_{0t}$ belong to $L_2(R)$ and depend continuously on t. For all $t \in [0, T]$, $\alpha - \alpha_0 \in W_2^2(R)$ so $\alpha - \alpha_0$ tends to zero as $|x| \to \infty$. Therefore the second integral in the left hand side vanishes. Furthermore, as

$$(\alpha-\alpha_0)(\alpha\alpha_x-\alpha_0\alpha_{0x})=\tfrac{1}{3}\frac{\partial}{\partial x}(\alpha-\alpha_0)^3+(\alpha-\alpha_0)^2\alpha_{0x}+\tfrac{1}{2}\alpha_0\frac{\partial}{\partial x}(\alpha-\alpha_0)^2$$

and, due to lemma 2 and the continuity in t, $\sup_{x,\tau\in H_t}|\alpha_{0x}(x,\tau)|<\infty$, we have for $t\in[0,T]$:

$$\left|\int_{-\infty}^{\infty}(\alpha-\alpha_0)(\alpha\alpha_x-\alpha_0\alpha_{0x})dx\right|=\tfrac{1}{2}\left|\int_{-\infty}^{\infty}\alpha_{0x}(\alpha-\alpha_0)^2dx\right|\leqq\tfrac{1}{2}R_0\|\alpha-\alpha_0\|^2,$$

where

$$R_0(t)=\sup_{x,\tau\in H_t}|\alpha_{0x}(x,\tau)|.$$

Using Schwarz's inequality we find

$$\left|\int_{-\infty}^{\infty}(\alpha-\alpha_0)\beta\alpha_x dx\right|\leqq R\|\beta\|\ \|\alpha-\alpha_0\|,$$

where

$$R(t)=\sup_{x,\tau\in H_t}\max[|\alpha_x(x,\tau)|,|\beta_x(x,\tau)|].$$

Now, we find for all $t\in[0,T]$:

$$\frac{d}{dt}\|\alpha-\alpha_0\|-\tfrac{1}{2}\varepsilon R_0\|\alpha-\alpha_0\|\leqq\mu\|(\alpha-\beta)_{xx}\|+\mu\|\alpha_{0xx}\|-\varepsilon\theta R\|\beta\|.$$

Multiplication of this inequality by $\exp[-\frac{1}{2}\varepsilon\int_0^t R_0(\tau)d\tau]$, integration with respect to t and use of: $R_0(t)\leqq R_0(t')$ when $t\leqq t'$, gives

$$\|(\alpha-\alpha_0)(t)\|\leqq\mu\int_0^t[\|(\alpha_{xx}-\beta_{xx})(\tau)\|+\|\alpha_{0xx}(\tau)\|-\mu^{-1}\varepsilon\theta R(\tau)\|\beta(\tau)\|]e^{\frac{1}{2}\varepsilon R_0(t)(t-\tau)}d\tau, \quad (1)$$

where $0\leqq t\leqq T$.

By now, our problem is reduced to finding estimates for R, R_0, $\|\beta\|$, $\|(\alpha-\beta)_{xx}\|$ and $\|\alpha_{0xx}\|$.

5. The Range of Validity of the SW Approximation

As we already noticed in the introduction, it will be quite interesting to compare T_m with T_{crit}, the time a shock wave starts to develop. It will turn out that T_{crit} for the solution of (3.1), …, (3.4) as well as for the solution of (3.5) and (3.6) is the same. It is defined as the smallest time at which the solution of the hyperbolic equation(s), obtained by putting $\mu=0$ in (3.5) ((3.1) and (3.2)), has (have) a vertical tangent. For $\mu=0$, we infer that

$$\alpha=\alpha_0=f(x-t-\varepsilon\alpha t), \qquad \beta=0.$$

Therefore

$$T_{\text{crit}}=\frac{1}{\varepsilon\sup_{x\in R}[-f'(x)]},$$

where the accent denotes differentiation with respect to x.

If $f(x)$ contains a compressive phase, i.e. $\sup_{x\in R}[-f'(x)]>0$, then T_{crit} is finite.

The main object of this section is the derivation of an expression for T_m in terms of ε, μ, θ and $f(x)$. We shall use *local* a priori estimates which also hold for $\mu=0$. Therefore, those estimates in which derivatives are involved, probably will not hold for times exceeding T_{crit} and with this method, we expect to find $T_m<T_{\text{crit}}$. However, the method is of interest as long as ε is so small that the sw approximation breaks down before $t=T_{\text{crit}}$.

Let us introduce some additional notations: $\alpha_x = r$, $\beta_x = s$, $\alpha_{xx} = p$, $\beta_{xx} = q$, $\alpha_{0x} = r_0$ and $\alpha_{0xx} = p_0$. We shall assume that r, s satisfy the ones-, p, q the twice- with respect to x differentiated partial differential equations and initial conditions (3.1), ..., (3.4), in the classical sense. Furthermore, let r_0 satisfy the ones-, p_0 the twice- with respect to x differentiated equations (3.5) and (3.6), in that sense.

Theorem 1. *Let, for all* $t \in [0, T]$, α, β *and* α_0 *belong to* $W_2^4(R)$ *and* $\gamma \geqq 0$. *Then, for*

$$0 \leqq t \leqq \frac{2\gamma}{5\varepsilon(1-\theta)\Lambda(f,\gamma)}, \tag{1}$$

where, for shortness,

$$\Lambda(f,\gamma) = \|f''\| \exp(\gamma) + \|f'\| \exp(\tfrac{1}{5}\gamma),$$

the following estimates hold

$$\begin{aligned}
&\|p(t)\|^2 + \|q(t)\|^2 \leqq \|f''\|^2 \exp(2\gamma), \\
&\|r(t)\|^2 + \|s(t)\|^2 \leqq \|f'\|^2 \exp(2\gamma/5), \\
&\|\alpha(t)\|^2 + \|\beta(t)\|^2 \leqq \|f\|^2 \exp[-\tfrac{4}{5}\theta\gamma/(1-\theta)], \\
&R(t) \leqq \|f''\| \exp(\gamma) + \|f'\| \exp(\gamma/5), \\
&\|p_0(t)\| \leqq \|f''\| \exp(\gamma), \\
&R_0(t) \leqq \|f''\| \exp(\gamma) + \|f'\| \exp(\gamma/5).
\end{aligned} \tag{2}$$

Proof. From (3.1), (3.2) and lemma 2, we easily infer that, for all $t \in [0, T]$, $\alpha_t \in L_2(R)$ and $\beta_t \in L_2(R)$. Upon multiplying (3.1) by α, (3.2) by β, integrating with respect to x from $-\infty$ to ∞ and adding the resulting equations we find:

$$\frac{d}{dt}(\|\alpha\|^2 + \|\beta\|^2) + 2\varepsilon\theta \int_{-\infty}^{\infty} \alpha\beta(r-s)\,dx = -2\mu \int_{-\infty}^{\infty} (r-s)^2\,dx.$$

The interchangement of differentiation with respect to t and integration with respect to x was allowed as α, β, α_t and β_t belong to $L_2(R)$ and depend continuously on t. Using Cauchy's inequality, we obtain

$$\|\alpha(t)\|^2 + \|\beta(t)\|^2 \leqq \|f\|^2 \exp[-2\varepsilon R(t)\theta t]. \tag{3}$$

In a similar way it is seen that:

$$\frac{d}{dt}(\|r\|^2 + \|s\|^2) + \varepsilon \int_{-\infty}^{\infty} [(r+\theta s)r^2 - (s+\theta r)s^2]\,dx = -2\mu \int_{-\infty}^{\infty} [p-q]^2\,dx,$$

and

$$\begin{aligned}
\frac{d}{dt}(\|p\|^2 + \|q\|^2) + \varepsilon \int_{-\infty}^{\infty} (5r+3\theta s)p^2\,dx - \varepsilon \int_{-\infty}^{\infty} (5s+3\theta r)q^2\,dx + \\
+ 2\varepsilon\theta \int_{-\infty}^{\infty} (r-s)pq\,dx = -2\mu \int_{-\infty}^{\infty} [p_x - q_x]^2\,dx.
\end{aligned}$$

From these relations, we infer:

$$\|r(t)\|^2 + \|s(t)\|^2 \leqq \|f'\|^2 \exp[\varepsilon R(t)(1-\theta)t], \tag{4}$$

$$\|p(t)\|^2 + \|q(t)\|^2 \leqq \|f''\|^2 \exp[5\varepsilon R(t)(1-\theta)t]. \tag{5}$$

According to lemma 2 and $[\sup_{x\in R} \max(|r|, |s|)]^2 \leqq 2(\sup_{x\in R}|r|)^2 + 2(\sup_{x\in R}|s|)^2$, we get

$$\sup_{x\in R} \max(|r|, |s|) \leqq (\|r\|^2+\|s\|^2)^{\frac{1}{2}} + (\|p\|^2+\|q\|^2)^{\frac{1}{2}},$$

or combining with (4) and (5)

$$R(t) \leqq \|f'\| \exp[\tfrac{1}{2}\varepsilon R(t)(1-\theta)t] + \|f''\| \exp[\tfrac{5}{2}\varepsilon R(t)(1-\theta)t]. \tag{6}$$

We have

$$e^{\gamma x} \leqq 1+(e^\gamma - 1)x \qquad (\gamma > 0), \tag{7}$$

holding for $0 \leqq x \leqq 1$.

Assume that

$$\tfrac{5}{2}\frac{\varepsilon R(t)(1-\theta)t}{\gamma} \leqq 1, \tag{8}$$

then we infer from (6) and (7)

$$R(t) \leqq \frac{\gamma(\|f'\|+\|f''\|)}{\gamma - \frac{5}{2}\varepsilon t(1-\theta)\{\Lambda(f,\gamma) - \|f'\| - \|f''\|\}}. \tag{9}$$

Assumption (8) must be satisfied. This implies that (1) must hold. Using (1), we find from (9) the inequality (2). From (8), (3), (4) and (5) and the remark that all estimates already found also hold for $\mu=\theta=0$, the remaining part of the theorem follows.

Remark

Instead of (7) we could have used: $e^x \leqq (1-x)^{-1}$ for $x \in [0, 1]$. However, this leads to a quite complicated algebraic equation of order three.

Theorem 2. *Let α, β and α_0 be defined as in the preceding theorem. If*

$$T_m = \max_{\gamma \geqq 0} \min[T_0(\gamma)/\varepsilon, \delta T_1(\gamma, \varepsilon)/\mu],$$

where

$$T_0(\gamma) = \frac{2\gamma}{5(1-\theta)\Lambda(f,\gamma)},$$

$$T_1(\gamma, \varepsilon) = \frac{\mu\|f\|\exp(-\frac{1}{5}\gamma)}{\mu(1+\sqrt{2})\|f''\|\exp(\gamma) - \varepsilon\theta\|f\|\Lambda(f,\gamma)\exp[\frac{2}{5}\theta\gamma/(\theta-1)]}$$

then the sw approximation may be called good.

Proof. As is easily seen, using theorem 1, for $t \leqq \min[T_0/\varepsilon, \delta T_1/\mu]$, (4.1) holds. As $\gamma \geqq 0$ is still arbitrary, we may choose this number such that $\min[T_0/\varepsilon, \delta T_1/\mu]$ assumes its maximum for some given ε, μ, θ and f, thus proving the theorem.

Corollary. We have

$$\frac{T_0(\gamma)}{\varepsilon} \leqq \frac{1}{\varepsilon}\max_{\gamma\geqq 0} \frac{2\gamma}{5(1-\theta)(\|f'\|+\|f''\|)\exp(\frac{1}{5}\gamma)},$$

and so, using the triangle inequality, lemma 2 and $\sup_{x\in R}|f'(x)| \geqq \sup_{x\in R}[-f'(x)]$, we obtain $T_m \leqq (\sqrt{2}/e)T_{\text{crit}}$. It is thus seen that $T_m < T_{\text{crit}}$ indeed.

Let ε_0 be a special value of ε for which $1+\frac{1}{2}\varepsilon(1+\theta)(\alpha+\beta) \geqq \delta < 0$ holds in H_T. Then the sound velocity a is real positive which is a necessary physical condition. As we assumed $\sup_{x,t\in H_T}|\alpha+\beta| \leqq 1$ (section 3), we may choose $\varepsilon_0 = (2-2\delta)/(1+\theta)$. Let $T_0(\gamma)$ assume its maximum for $\gamma=\gamma_m$. As may be easily verified $\gamma_m = \gamma_m(\|f'\|/\|f''\|)$ and $1 \leqq \gamma_m \leqq 5$. Define ε_1 by

$$\frac{T_0(\gamma_m)}{\varepsilon_1} = \frac{\delta T_1(\gamma_m, \varepsilon_1)}{\mu}.$$

ε_1 may be infinite and even negative. Let γ_1, γ_2 and γ_3 satisfy

$$\frac{T_0(\gamma)}{\varepsilon} = \frac{\delta T_1(\gamma, \varepsilon)}{\mu}$$

and $\gamma_3 \leqq \gamma_2 \leqq \gamma_1$. If $\varepsilon_1 \geqq 0$, then define $\varepsilon_2 = \min(\varepsilon_0, \varepsilon_1)$, else $\varepsilon_2 = \varepsilon_0$.

Theorem 3. *Let* α, β *and* α_0 *be defined as in theorem* 1. *The sw approximation is good*
(i) *for* $0 \leqq \varepsilon \leqq \varepsilon_2$ *if* $T_m = \delta T_1(\gamma_3, \varepsilon)/\mu$,
(ii) *for* $\varepsilon_2 < \varepsilon \leqq \varepsilon_0$ *(assuming this interval is not empty) if* $T_m = T_0(\gamma_m)/\varepsilon$.
If $\delta \|f\| \, \|f'\|_1 / \|f''\| \ll 1$ *is satisfied, we have* $\varepsilon_2 \gg \mu$.

Proof. Let $0 \leqq \varepsilon \leqq \varepsilon_2$, then $T_0(\gamma)/\varepsilon$ and $\delta T_1(\gamma, \varepsilon)/\mu$, considered as functions of γ, may have three points of intersection and

$$\min[T_0/\varepsilon, \delta T_1/\mu] = \begin{cases} T_0(\gamma)/\varepsilon & (0 \leqq \gamma \leqq \gamma_3), \\ \delta T_1(\gamma, \varepsilon)/\mu & (\gamma_3 \leqq \gamma \leqq \gamma_2), \\ T_0(\gamma)/\varepsilon & (\gamma_2 \leqq \gamma \leqq \gamma_1). \end{cases}$$

For $0 \leqq \gamma \leqq \gamma_m$, $T_0(\gamma)$ is a monotonically increasing-, for $\gamma \geqq \gamma_m$ a monotonically decreasing function. As $T_1(\gamma, \varepsilon)$ is monotonically decreasing with respect to γ, (i) follows.

Let $\varepsilon_2 < \varepsilon \leqq \varepsilon_0$ and assume this interval is not empty. Then $T_0(\gamma)/\varepsilon$ and $\delta T_1(\gamma, \varepsilon)/\mu$ only intersect for $\gamma > \gamma_m$.

This proves (ii).

Finally, using $1 \leqq \gamma_m \leqq 5$ and $-1 < \theta \leqq 0$, we obtain for $\varepsilon_1 \in [0, \varepsilon_0]$:

$$\varepsilon_2 \geqq \frac{\mu \|f''\|}{5\delta \|f\| \, \|f'\|_1},$$

from which the remaining part of the theorem immediately follows.

For $\varepsilon_2 < \varepsilon \leqq \varepsilon_0$, the upper bound T_m, [see (ii)], is essentially due to the method followed (see also the corollary after theorem 2). If $0 \leqq \varepsilon \leqq \varepsilon_2$ then that bound results from the coupling between the α- and β-mode and T_m is a monotonically decreasing function of ε and μ. This agrees with what we expected from a physical point of view.

In case the equations are linear, i.e. $\varepsilon = 0$, we find:

$$T_m = \frac{\delta \|f\|}{(1+\sqrt{2})\mu \|f''\|}.$$

Furthermore, when the ratio $\|f''\|/\|f\|$ decreases and ε is small enough, T_m increases. This happens when we start at $t=0$ with a wave packet with a larger dominant wave length.

If $f(x)$ satisfies $\delta \|f\| \|f'\|_1 / \|f''\| \ll 1$, a condition which could be expressed by: "$f(x)$ should not vary *too* slowly", then $\varepsilon_2 \gg \mu$ and the method followed is useful for a large range of ε-values.

One may ask whether it is possible to improve the results found by using other types of estimates. We remark that we have not taken advantage of the dissipative terms in equations (3.1), (3.2) and (3.5). Therefore, especially when $\varepsilon \in [\varepsilon_2, \varepsilon_0]$, it may be even possible to prove that the sw approximation holds for times exceeding T_{crit}.

In the next section, some *global* a priori estimates have been constructed for the solution of Cauchy's problem for the Burgers equation. Unfortunately, this has not been possible for the solution of (3.1), ..., (3.4). However, with help of the global estimates found, we may estimate the second term in (4.1). This gives at least some indication whether or not, in this way it will be possible to improve the results found in this section.

6. A Priori Estimates for Burgers' Equation

First, we shall study the mixed problem for the equation of Burgers:

$$\alpha_{0t}+[1+\varepsilon\alpha_0]\alpha_{0x}=\mu\alpha_{0xx},$$
$$\alpha_0(x,0)=\chi_N(x)f(x) \qquad (|x|\leqq N),$$
$$\alpha_0(N,t)=\alpha_0(-N,t)=0 \qquad (0\leqq t\leqq T),$$

where $\chi_N(x)$ is a sufficiently smooth function such that $0\leqq\chi_N\leqq 1$, $\chi_N=1$ for $|x|\leqq N-\sqrt{N}-1$, $\chi_N=0$ for $|x|\geqq N$, $\chi_N'(\pm N)=\chi_N''(\pm N)=0$ and the derivatives of χ_N are uniformly bounded with respect to N. $f(x)$ is defined as in the preceding sections.

Extend the definition of $f(x)$ to Q_T by putting $f(x,t)=f(x)$ for all $x,t\in Q_T$. Let $f(x,t)\in C^{2+\nu}$-(Q_T), then, by a suitable choice of χ_N, $(\chi_N f)(x,t)\in C^{2+\nu}(Q_T)$ and the compatibility condition $([1+\varepsilon\alpha_0(x,0)]\alpha_{0x}(x,0)=\mu\alpha_{0xx}(x,0))$ at $x=\pm N$ is satisfied. So according to Oleinik and Kruzhkov [12], a unique solution $\alpha_0(x,t)\in C^{2+\nu}(Q_T)$ exists for all $T>0$. In the following, we shall denote this solution by α_0^N.

As a direct consequence of the generalized maximum principle for parabolic equations (see ch. I., section 2 of [11]), we have, for all $T>0$:

$$\sup_{x,t\in Q_T}|\alpha_0^N(x,t)|\leqq\sup_{|x|\leqq N}|f(x)|.$$

Next, we state:

Lemma 3. *For all* $T>0$:

$$\sup_{x,t\in Q_T}|\alpha_{0x}^N(x,t)|\leqq\max\left\{\tfrac{9}{2}e^2\frac{\varepsilon}{\mu}\left[\sup_{|x|\leqq N}|f|\right]^2,\ \sup_{|x|\leqq N}[(\chi_N f)']+\frac{\varepsilon}{\mu}\left[\sup_{|x|\leqq N}|f|\right]^2+\frac{1}{\mu}\left[\sup_{|x|\leqq N}|f|\right]\right\}.$$

The proof may be given by using the method of auxiliary functions due to Bernshtein (cf. [12]). It is postponed to the appendix.

Now, we return to the Cauchy problem.

Theorem 4. *The solution* $\alpha_0(x,t)$ *of* (3.5) *and* (3.6) *where* $f\in C^{2+\nu}(H_T)$, *for all* $T<0$, *exists, is unique and belongs to* $C^{2+\nu}(H_T)$. *Furthermore, for all* $T<0$:

$$\sup_{x,t\in H_T}|\alpha_0(x,t)|\leqq\sup_{x\in R}|f(x)|\leqq 1,$$

$$\sup_{x,t\in H_T}|\alpha_{0x}(x,t)|\leqq\max\left\{\sup_{x\in R}|f'(x)|+\frac{\varepsilon}{\mu}\left[\sup_{x\in R}|f|\right]^2+\frac{1}{\mu}\left[\sup_{x\in R}|f|\right],\right.$$
$$\left.\tfrac{9}{2}e^2\frac{\varepsilon}{\mu}\left[\sup_{x\in R}|f(x)|\right]^2\right\}\leqq\max\left[1+\frac{1}{\mu}+\frac{\varepsilon}{\mu},\ \tfrac{9}{2}e^2\frac{\varepsilon}{\mu}\right]. \qquad (1)$$

Proof. The existence and uniqueness follow immediately from theorem 8.1, p. 495 of [11]. Let us consider α_0^N for $N>N_0$ in a fixed cylinder $Q_T^{N_0}$. Then [11] it is shown that a subsequence $\{\alpha_0^{N_k}\}$ exists that converges together with the derivatives $\alpha_{0x}^{N_k}$, $\alpha_{0xx}^{N_k}$ and $\alpha_{0t}^{N_k}$ to the solution α_0 of the Cauchy problem (3.5) and (3.6) and the corresponding derivatives in any fixed $Q_T^{N_0}$. Now, choose $\chi_N(x)$ such that $|\chi_N'(x)|\leqq c/(\sqrt{N}+1)$ (c a positive constant). Then, for any positive numbers ε and ε', a number $N_1(\varepsilon,\varepsilon')$ can be found such that for $N_k>N_1(\varepsilon,\varepsilon')$ and

$$\sup_{|x|\leqq N_k}|(f\chi_{N_k})'|\leqq\sup_{x\in R}|f'|+\varepsilon,$$

$$\sup_{Q_T^{N_0}} |\alpha_{0x}| \leqq \sup_{Q_T^{N_0}} |\alpha_{0x}^{N_k}| + \varepsilon' \leqq \sup_{Q_T^{N_k}} |\alpha_{0x}^{N_k}| + \varepsilon' \leqq \varepsilon + \varepsilon' +$$

$$+ \max\left\{ \sup_{x\in R} |f'(x)| + \frac{\varepsilon}{\mu}\left[\sup_{x\in R} |f(x)|\right]^2 + \frac{1}{\mu}\left[\sup_{x\in R} |f|\right], \tfrac{9}{2}e^2 \frac{\varepsilon}{\mu}\left[\sup_{x\in R} |f(x)|\right]^2 \right\}.$$

As ε, ε' and N_0 are arbitrary we immediately deduce (1). The proof of the remaining inequality runs along similar lines.

Remark

To show that the estimate obtained for $|\alpha_{0x}|$ is quite accurate, we remark that the front of a shock wave solution of Burgers' equation, when fully developed, is approximately described by a steady state solution of that equation (Murray [13], Lighthill [4]). That is by

$$\alpha_0 = \tfrac{1}{2}(a_1 + a_2) + \tfrac{1}{2}(a_1 - a_2) \tanh \left\{ \frac{\varepsilon(a_1 - a_2)[x - t - \frac{1}{2}\varepsilon(a_1 + a_2)t]}{4\mu} \right\},$$

where a_1 (a_2) is the value of α_0 immediately behind (before) the front of the shock wave. Differentiation of this expression with respect to x shows that the result obtained has the same order of magnitude as the second term between curly brackets in (1).

Finally, we prove

Theorem 5. *Let* $f \in C^{2+\nu}(H_T)$, p_0, *defined as in section 5, satisfy the twice with respect to x differentiated equations* (3.5) *and* (3.6) *in the classical sense and let, for all* $t \in [0, T]$, α_0 *belong to* $W_2^4(R)$. *Then, for* $t \in [0, T]$:

$$\|p_0(t)\| \leqq \|f''\| \exp\left(\tfrac{25}{4}\frac{\varepsilon^2 t}{\mu}\right).$$

Proof. According to the preceding theorem $\alpha_0 \in C^{2+\nu}(H_T)$ for every $T > 0$. Differentiate (3.5) with respect to x twice. For all $t \in [0, T]$, $P_{0t} \in L_2(R)$ as may be seen from the resulting equation easily. Multiply that equation by p_0 and integrate over the entire x-axis. Then, using lemma 1, partial integration with respect to x and interchanging differentiation with respect to t and integration (cf. theorem 1), we find:

$$\frac{d}{dt}\|p_0\|^2 - 10\,\varepsilon \int_{-\infty}^{\infty} \alpha_0 p_0 p_{0x}\, dx = -2\mu \int_{-\infty}^{\infty} p_{0x}^2\, dx\,.$$

According to Cauchy's inequality

$$\int_{-\infty}^{\infty} \alpha_0 p_0 p_{0x}\, dx \leqq \left\{\sup_{x\in R} |\alpha_0|\right\}\left\{\frac{\nu}{2}\|p_0\|^2 + \frac{1}{2\nu}\|p_{0x}\|^2\right\} \qquad (\nu < 0)\,.$$

Then, using theorem 4, we obtain

$$\frac{d}{dt}\|p_0(t)\|^2 - 5\varepsilon\nu\|p_0(t)\|^2 \leqq \left(-2\mu + \frac{5\varepsilon}{\nu}\right)\|p_{0x}(t)\|^2\,.$$

Choosing $\nu = 5\varepsilon/2\mu$, we deduce the theorem.

Using the last two theorems, we infer from (4.1) that

$$\|(\alpha - \alpha_0)(t)\| \leqq \mu \int_0^t [\|(\alpha_{xx} - \beta_{xx})(\tau)\| - \mu^{-1}\varepsilon\theta R(\tau)\|\beta(\tau)\|]\, e^{\frac{1}{2}\varepsilon R_0(t)(t-\tau)}\, d\tau +$$

$$+ \left\{ \mu t \|f''\| \exp\left[\tfrac{1}{2}\varepsilon R_0(t)t + \frac{25}{4}\frac{\varepsilon^2 t}{\mu}\right]\right\}.$$

Now, for the terms between curly brackets to be smaller than $\delta\|f\|$, it is necessary that

$$t \leqq \min\left\{\frac{\delta\|f\|}{\mu\|f''\|}, \quad \left[\frac{\delta\|f\|}{\|f''\|(\frac{1}{2}\varepsilon\mu R_0(t)+\frac{25}{4}\varepsilon^2)}\right]^{\frac{1}{2}}\right\}. \tag{2}$$

Assume that $\delta\|f\|\ \|f'\|_1/\|f''\| \ll 1$. Then $\varepsilon_2 \gg \mu$ and according to the former section we do not expect T_m to be larger than T_{crit}. This is confirmed by the method followed in this section as is easily seen from (2). However, if $\delta\|f\|\ \|f'\|_1/\|f''\| \ll 1$ is violated, it does seem possible that $T_m \geqq T_{\text{crit}}$ for some initial condition. Therefore, I think future investigations should be concerned with a priori estimates for the set of nonlinear equations. This is not easy, for the nonlinear set (3.1) and (3.2) is not purely parabolic. It might be termed a mixed parabolic-hyperbolic set.

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Appendix

We shall give a proof of lemma 3.

First, we prove that for all $T \geqq 0$:

$$\max_{\Gamma_T}|\alpha^N_{0x}(x,t)| \leqq \max_{|x|\leqq N}|(f(x)\chi_N(x))'| + \frac{\varepsilon}{\mu}\left[\max_{x\leqq N}|f(x)|\right]^2 + \frac{1}{\mu}\left[\max_{|x|\leqq N}|f(x)|\right]. \tag{1}$$

Define

$$v = \alpha^N_0 + M\exp[-K(x+N)], \tag{2}$$

where $K \geqq k > 0$, $M \geqq m > 0$ but further, as yet, arbitrary. Substitution of (2) in (3.5) and putting $K = \frac{1}{\mu} + \frac{\varepsilon}{\mu}\max_{|x|\leqq N}|f|$, we obtain: $-v_t - v_x - \varepsilon\alpha_0^{N v_x} + \mu v_{xx} > 0$. Therefore v cannot assume a positive maximum in Q_T/Γ_T. Since, when we choose $M = K^{-1}\max_{|x|\leqq N}|(f\chi_N)'| + \max_{|x|\leqq N}|f|$, $v(t,x)$ assumes its greatest value for $x=-N$, then, $v_x \leqq 0$ for $x=-N$ and therefore $\alpha^N_{0x}|_{x=-N} \leqq MK$. By considering $\alpha^N_0 - M\exp[-K(x+N)]$ we find similarly that $\alpha^N_{0x}|_{x=-N} \geqq -MK$. Thus, we have an estimate for $|\alpha^N_{0x}|$ at $x=-N$ and similarly for $x=N$. We find (1).

Next, we prove the remaining part of the lemma. Substitute the unknown function $\alpha^N_0 = \phi(v)$, $\phi'(v) \geqq \phi_0 > 0$ in (3.5). Then, we obtain

$$v_t + [1+\varepsilon\phi]v_x - \mu v_{xx} - \mu\frac{\phi''}{\phi'}v_x^2 = 0.$$

Differentiation of this equation with respect to x is allowed according to theorem 9 of Oleinik and Kruzhkov [12]. Therefore, the function $p = v_x$ satisfies

$$p_t = -(1+\varepsilon\phi)p_x - \varepsilon\phi' p^2 + \mu p_{xx} + 2\mu\frac{\phi''}{\phi'}pp_x + \mu\left(\frac{\phi''}{\phi'}\right)' p^3.$$

At a maximum of $|p|$ in Q_T/Γ_T we have $p_x=0$, $-pp_{xx} \geqq 0$ and $pp_t \geqq 0$. So we find

$$0 \leqq -\varepsilon\phi' p^3 + \mu\left(\frac{\phi''}{\phi'}\right)' p^4 \tag{3}$$

Now, choose

$$\phi(v) = -2M + 3eM\int_0^v \exp(-s^m)ds \qquad (m>0), \tag{4}$$

where $M = \max_{|x| \leqq N} |f|$.

If α_0^N varies in the interval $[-M, M]$, v varies over a finite interval $[v_1, v_2]$. Since

$$\int_0^{v_1} e^{-s^m} ds = \frac{1}{3e} > \int_0^{1/(3e)} e^{-s^m} ds\,, \quad \int_0^{v_2} e^{-s^m} ds = \frac{1}{e} < \int_0^1 e^{-s_m} ds\,,$$

we obtain

$$\frac{1}{3e} < v_1 < v_2 < 1\,. \tag{5}$$

Now, using (3), (4) and (5), it is seen that

$$|p| \leqq \frac{\varepsilon}{\mu} \frac{3eMe^{-v^m}}{m(m-1)v^{m-2}}.$$

The right hand side of this inequality approximately assumes its smallest value for $m=2$. Putting $m=2$ and using (5) once again, we find, for all $T \geqq 0$

$$\max_{Q_T/\Gamma_T} |\alpha_{0x}^N| \leqq \tfrac{9}{2} e^2 \frac{\varepsilon}{\mu} M^2\,. \tag{6}$$

If $|\alpha_{0x}^N|$ does not assume a maximum in Q_T/Γ_T, then

$$\max_{Q_T/\Gamma_T} |\alpha_{0x}^N| \leqq \max_{\Gamma_T} |\alpha_{0x}^N(x, t)|\,. \tag{7}$$

Combining (1), (6) and (7), we find the lemma.

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